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THE DISSIPATIVE EVOLUTION OF AN ALFVEN SOLITON†

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A model equation for non-linear Alfven waves, allowing for dispersion and dissipation in magnetohydrodynamics, is derived. The evolution of an Alfven soliton is examined. © 1999 Elsevier Science Ltd. All rights reserved.

1. The initial system is taken to consist of the one-dimensional magnetohydrodynamic equations with Hall dispersion and dissipation, represented by the viscosity and magnetic viscosity. All the quantities are assumed to depend only on the variables x and t. In dimensionless form, this system is [1]

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial s}\frac{\partial s}{\partial x}-\frac{1}{2}\frac{\partial}{\partial x}\left|B^2\right|+\frac{4}{3}\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}\right)=B_x\frac{\partial B_y}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}\right)=B_x\frac{\partial B_z}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 w}{\partial x^2}$$

$$\rho T\left(\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}\right)=\frac{4}{3}\frac{1}{\mathrm{Re}}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{4\pi R_m}\left(\left(\frac{\partial B_z}{\partial x}\right)^2+\left(\frac{\partial B_y}{\partial x}\right)^2\right)+\frac{1}{\mathrm{Re}}\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$\frac{\partial B_y}{\partial t}=-\frac{\partial}{\partial x}\left(uB_y-vB_x\right)-\frac{k}{\rho}\left(-\frac{\partial^2 B_z}{\partial x^2}+\frac{1}{\rho}\frac{\partial B_z}{\partial x}\frac{\partial \rho}{\partial x}\right)+\frac{1}{R_m}\frac{\partial^2 B_y}{\partial x^2}$$

$$\frac{\partial B_z}{\partial t}=\frac{\partial}{\partial x}\left(wB_x-uB_z\right)-\frac{k}{\rho}\left(\frac{\partial^2 B_y}{\partial x^2}-\frac{1}{\rho}\frac{\partial B_y}{\partial x}\frac{\partial \rho}{\partial x}\right)+\frac{1}{R_m}\frac{\partial^2 B_z}{\partial x^2}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0$$

$$B_x=\mathrm{const},\quad k=\frac{cm_iB_x}{4\pi e}$$

Here u, v and w are the components of the velocity vector, B_x, B_y, B_z are the components of the magnetic induction vector, p is the pressure, s is the entropy, T is the temperature, Re is the Reynolds number, R_m is the magnetic Reynolds number, and k is the dimensionless Hall parameter.

We shall confine ourselves below to long waves, so that we can immediately introduce a small parameter δ (δ is of the same order of smallness as the wave number). We must also assume that the dissipation is small, of the form

$$\delta\sim\frac{1}{\mathrm{Re}}+\frac{1}{R_m}$$

Another important point is that the dispersion is assumed to be finite, that is, $k\sim l$. Moreover, the wave propagation must not be longitudinal: $\sin\alpha\sim 1$, where α is the angle between the x axis and the direction of the unperturbed magnetic field. We represent B_y and B_z in the form

$$B_y=b\sin\theta,\quad B_z=b\cos\theta$$

where b is the magnitude of the transverse component of the magnetic field and θ is the direction of the magnetic field in the (y, z) plane.

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We replace the independent variables according to the formulae

$$\xi = \delta(x - t\cos\alpha), \quad \tau = \delta^3 t$$

Then, using the technique described in [2], expanding all the variables in powers of δ and then substituting into the initial system, we obtain the equation

$$\frac{\partial f}{\partial \tau} + \frac{3}{2}\beta f^2 \frac{\partial f}{\partial \xi} + \beta \frac{\partial^3 f}{\partial \xi^3} - \gamma \frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \xi} + f \int_{-\infty}^{\xi} (f(\lambda))^2 d\lambda\right) = 0 \tag{1.1}$$

$$f = \frac{\partial \theta_0}{\partial \xi}, \quad \beta = -\frac{k^2}{2\sin^2\alpha}\left(\frac{a^2}{\cos\alpha} - \cos\alpha\right), \quad \gamma = \frac{1}{2\delta}\left(\frac{1}{\mathrm{Re}} + \frac{1}{R_m}\right)$$

where θ_0 is the constant term in the expansion of θ in powers of δ, and a is the unperturbed velocity of sound.

This equation is more accurate than that obtained in [3] and is an extension of the well-known equations for Alfven waves which allow only for dissipation and only for dispersion respectively. The cases of pure dispersion [4] and pure dissipation [5, 6] have been discussed before.

Now, as in [7], we confine ourselves to the case $\gamma/\beta = \varepsilon$, where ε is a small parameter (dispersion predominates over dissipative effects).

We make the replacement of variables

$$\beta\tau = t, \; f = 2u, \; \xi = x$$

In this case Eq. (1.1) reduces to the form

$$\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \varepsilon \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + 4u \int_{-\infty}^{x} (u(y))^2 dy\right) = 0 \tag{1.2}$$

and is the perturbed modified Korteweg-de-Vries (MKdV) equation.

It should be borne in mind below that the variables u, x and t we are using here are not the same as the physical variables.

2. As we know [7], the perturbed MKdV equation

$$\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \varepsilon R[u] \tag{2.1}$$

is equivalent to the operator equation

$$i\frac{\partial L}{\partial t} + [L, A] = i\varepsilon R \tag{2.2}$$

where L and A are linear operators which depend on u and a $[L, A]$ is a commutator; the eigenvalues of L are independent of time, and the perturbation operator R (for real equations) is the same as R on the right-hand side of (2.1).

Since Eq. (2.2) is exact, it can be used to develop the approximate theory of perturbations for MKdV solitons [8]. Here we consider the so-called adiabatic approximation of that theory.

The solution of Eq. (2.1) can be represented in the form

$$u_s(x, t) = 2\upsilon(t)\operatorname{sech} h(z) + w(x, t), \quad z = 2\upsilon(t)\,(x - \mu(t)) \tag{2.3}$$

where the first term defines the evolution of a soliton nucleus while the second describes the development of the tail. In the given approximation, we neglect the function $w(x, t)$ and can thus determine only the functions $\upsilon(t)$ and $\mu(t)$. Thus, from a physical point of view, the approximation is only applicable for small t for which a growing tail has no significant influence on the type of motion. In the case when $\upsilon = \upsilon_0$, $\mu = 4\upsilon_0^2 t$ formula (2.3) describes a soliton of the unperturbed MKdV equation.

The basic formulae of the adiabatic approximation have the form [8]

$$\frac{d\upsilon}{dt} = \frac{\varepsilon}{2} \int_{-\infty}^{+\infty} \frac{R[u_s(z)]}{\mathrm{ch}(z)} dz, \quad \frac{d\mu}{dt} = \frac{\varepsilon}{4\upsilon^2} \int_{-\infty}^{+\infty} \frac{zR[u_s(z)]}{\mathrm{ch}(z)} dz + 4\upsilon^2 \tag{2.4}$$

3. We will investigate Eq. (1.2) using perturbation theory. The perturbation operator for Eq. (1.2) has the form

$$R[u]=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+4u\int\limits_{-\infty}^{x}(u(y))^2dy\right) \tag{3.1}$$

Substituting (2.3) into (3.1), we obtain

$$R[u_s]=8\upsilon^3\frac{6-3\mathrm{ch}^2(z)-4\mathrm{ch}(z)\mathrm{sh}(z)}{\mathrm{ch}^3(z)} \tag{3.2}$$

When (3.1) and (3.2) are substituted into the right-hand sides of (2.4), the latter reduce to improper integrals which, when evaluated, give

$$\frac{d\upsilon}{dt}=8\upsilon^3\varepsilon,\quad \frac{d\mu}{dt}=4\upsilon^2-8\upsilon\varepsilon \tag{3.3}$$

After integrating we obtain

$$\upsilon(t)=\frac{\upsilon_0}{\sqrt{1-16\varepsilon\upsilon_0^2t}},\quad \mu(t)=-\frac{1}{4\varepsilon}\ln\left(1-16\varepsilon\upsilon_0^2t\right)+\frac{1}{\upsilon_0}\left(\sqrt{1-16\varepsilon\upsilon_0^2t}-1\right) \tag{3.4}$$

These formulae for small t can be used to follow the change of velocity and amplitude of an Alfven soliton under the effect of dissipation.

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